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An analytical solution to coupled heat and moisture diffusion transfer in porous materials

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Abstract

An analytical method is proposed for solving the problem of coupled heat and moisture transfer in porous materials. The coupled partial differential equations and boundary conditions are first subjected to Laplace transformation, the equations are reduced to ordinary differential equations, then the equations are converted into a single fourth-order ordinary differential equation by introducing a transformation function. The solution of the equation can be easily obtained, and thus, the temperature and moisture distributions in the transform domain can be determined. Finally, the transformed values are analytically or numerically inverted to obtain the time domain results. Therefore, the transient solution at any given time can be evaluated. The results are identical with published analytical solutions for a special case using decoupling technique, and they agree with a published analytical solution for wood slab. The method is compact enough to be generally applied to problems of heat and mass transfer in porous media. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

Simultaneous heat and moisture transfer accompanied by phase change and/or absorption heat in some porous and composite materials is a process which occurs frequently in various engineering applications. The moisture movement contributes to heat transfer, while phase-change and/or heat of absorption within the material act as heat sources or sinks.

The coupled system for temperature and moisture potential can be handled through both analytical and numerical approaches, depending on complexities of the specific problem considered. For linear problems, the analytical solutions, based on classical integral transform approach, have been obtained by Mikhailov et al. [1]. It was later discovered by the same group [2] that the existence of complex eigenvalues, not accounted for in the solutions previously reported, could significantly alter the temperature and moisture distributions. Liu et al. [3] later reconfirmed such findings by discovering a pair of complex eigenvalue. Recently, Mikhailov et al. [4] again found another complex eigenvalue. In the cited papers, there exist numerical difficulties in computing the complex conjugate eigenvalues that lead to an incorrect solution containing high frequency oscillation of limited usefulness. Chang et al. [5] applied a decoupling technique to coupled governing equations, but failed to address the case of simultaneous coupling of governing equations and boundary conditions. Ribeiro et al. [6] treated the coupled system by applying the generalized integral transform technique with complicated procedures. Recently, Cheroto et al. [7] presented a modified lumped system analysis method to yield approximate

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Nomenclature

c_m moisture capacity (kg/kg °M)
c_p heat capacity (J/kg K)
C moisture content (kg/kg)
D equivalent diffusion coefficient of moisture content (m^2/s)
D_m conductivity coefficient of moisture content (kg/m s °M)
h_C convective heat transfer coefficient (W/m^2 K)
h_m convective moisture transfer coefficient (kg/m^2 s °M)
h_{LV} heat of phase change (kJ/kg)
I_0 modify Bessel function of the first kind of order zero
I_1 modify Bessel function of the first kind of order first
J_0 Bessel function of the first kind of order zero
J_1 Bessel function of the first kind of order first
k thermal conductivity (W/m K)
K_0 modify Bessel function of the second kind of order zero
K_1 modify Bessel function of the second kind of order first
ℓ half thickness of specimen (m)
L equivalent diffusion coefficient of temperature (m^2/s)
m moisture potential (°M)
m_0 initial moisture potential (°M)
$\bar{m}$ Laplace transformation of m
S Laplace transformation parameter
t time (s)
T temperature (K)
T_0 initial temperature (K)
$\bar{T}$ Laplace transformation of T
Y_0 Bessel function of the second kind of order zero
Y_1 Bessel function of the second kind of order first

Greek symbols
ρ material density (kg/m^3)
ν coupling coefficient due to moisture migration (K/°M)
λ coupling coefficient due to heat conduction (°M/K)
ϕ transformation function
γ heat of absorption or desorption (kJ/kg)
ε ratio of vapor diffusion coefficient to coefficient of total moisture diffusion

Subscripts
0 initial condition
m moisture
∞ ambient atmosphere

solutions, avoiding difficulties experienced by Mikhailov et al., Liu et al., and Ribeiro et al., but sacrificing accuracy.

In this paper, a different analytical approach is developed. The coupled system is first subjected to Laplace transformation. The coupled partial differential equations thus are reduced to the coupled ordinary differential equations. Then, the temperature and moisture in the transformed domain are expressed in terms of a transformation function. Consequently, the system of ordinary differential equations can be reduced to a single fourth-order ordinary differential equation. The solution of the equation can be obtained and the coefficients are determined from the boundary conditions. Finally, the transformed solution can be analytically or numerically inverted to yield the time domain results.

2. Problem formulation

A typical heat and mass transfer problem is governed by Luikov's equations [8], which relate to drying a porous moist slab under constant pressure. The phase-change occurring within the slab act as heat source or sink resulting in the coupled relationship between mass transfer and heat transfer. In a coupled problem, the heat of absorption or desorption is generally one of the sources or sinks as well. This heat is not negligible for some hygrothermal materials [9,10]. In the present study, one-dimensional governing equations with coupled temperature and moisture for a porous slab are considered, and the effect of the absorption or desorption heat is added. Material properties and pressure are considered to be constant throughout the material. A local thermodynamic equilibrium between the fluid and the porous matrix is assumed. Moreover, the coupled equations can be generalized to apply to cases of hollow cylindrical and hollow spherical geometries. The equations are

$$\rho c_p x^{1-2n}\frac{\partial T}{\partial t} = k\frac{\partial}{\partial x}\left(x^{1-2n}\frac{\partial T}{\partial x}\right) + \rho c_m(\varepsilon h_{LV} + \gamma)x^{1-2n}\frac{\partial m}{\partial t} \tag{1a}$$

$$\rho c_m x^{1-2n}\frac{\partial m}{\partial t} = D_m\frac{\partial}{\partial x}\left(x^{1-2n}\frac{\partial m}{\partial x}\right) + D_m\delta\frac{\partial}{\partial x}\left(x^{1-2n}\frac{\partial T}{\partial x}\right) \tag{1b}$$

where $n = 1/2$ for slab, $n = 0$ for hollow cylinder, and $n = -1/2$ for hollow sphere. T is the temperature, and m is the moisture potential, k and D_m are the thermal and moisture conductivity coefficients, respectively, c_p and c_m are the heat and moisture capacities of the medium, respectively, ρ is the material density, h_{LV} is the heat of evaporative phase-change, γ represents the heat of absorption or desorption, δ is the thermogradient coefficient, and ε is the ratio of the vapor diffusion coefficient to the coefficient of total moisture diffusion. All the material properties mentioned above are effective properties. The moisture potential m is related to the moisture content C, and

$$C = c_m m \tag{2}$$

The coupling diffusion system represented by Eqs. (1a) and (1b) contains not only general diffusion equations, but also some source or sink terms. The governing equation (1a) expresses the balance of thermal energy within the body; the last term in this equation represent the heat sources or heat sinks due to liquid-to-vapor phase-change and to the heat of absorption or desorption. Similarly, Eq. (1b) expresses the balance of moisture within the medium; the last term in this equation represents the moisture source or moisture sink with respect to the temperature gradient.

To simplify the notation, dividing Eq. (1a) by $\rho c_p x^{1-2n}$ and using the expression for $\partial/\partial x(x^{1-2n}\partial T/\partial x)$ obtained from Eq. (1a) into Eq. (1b), then rearranging the two new equations, yields

$$Lx^{2n-1}\frac{\partial}{\partial x}\left(x^{1-2n}\frac{\partial T}{\partial x}\right) = \frac{\partial T}{\partial t} - \nu\frac{\partial m}{\partial t} \tag{3a}$$

$$Dx^{2n-1}\frac{\partial}{\partial x}\left(x^{1-2n}\frac{\partial m}{\partial x}\right) = \frac{\partial m}{\partial t} - \lambda\frac{\partial T}{\partial t} \tag{3b}$$

where

$$L = \frac{k}{\rho c_p} \tag{4a}$$

$$D = \frac{kD_m}{\rho c_m\left[k + D_m\delta(\varepsilon h_{LV} + \gamma)\right]} \tag{4b}$$

$$\nu = \frac{c_m(\varepsilon h_{LV} + \gamma)}{c_p} \tag{4c}$$

$$\lambda = \frac{c_p D_m \delta}{c_m\left[k + D_m\delta(\varepsilon h_{LV} + \gamma)\right]} \tag{4d}$$

In Eqs. (3a) and (3b), ν and λ are positive coupling coefficients due to moisture migration and heat conduction, respectively; L and D are also always positive expressing the equivalent temperature diffusion coefficient and the equivalent moisture diffusion coefficient, respectively. The moisture in Eq. (3a) will play the role of a heat source for the temperature distribution, if the moisture rate is positive (i.e. $\partial m/\partial t > 0$), and act as a heat sink if the moisture rate is negative (i.e. $\partial m/\partial t < 0$). Similarly, the temperature may play the role of a moisture source or a moisture sink, depending on the temperature rate being positive or negative. Therefore, the coupling diffusion system rewritten as Eqs. (3a) and (3b) more compactly and clearly represents the same physical process modeled by Eqs. (1a) and (1b).

At the boundaries of the domain, the latent heat of vaporization becomes part of the energy balance, and the mass diffusion caused by the temperature and moisture gradients affects the mass balance. The associated hygrothermal boundary and initial conditions are

$$k\frac{\partial T(x_1, t)}{\partial x} = h_{C1}\left[T(x_1, t) - T_{\infty 1}\right] + (1-\varepsilon)h_{LV}h_{m1}\left[m(x_1, t) - m_{\infty 1}\right] \tag{5a}$$

$$-k\frac{\partial T(x_2, t)}{\partial x} = h_{C2}\left[T(x_2, t) - T_{\infty 2}\right] + (1-\varepsilon)h_{LV}h_{m2}\left[m(x_2, t) - m_{\infty 2}\right] \tag{5b}$$

$$D_m\frac{\partial m(x_1, t)}{\partial x} + D_m\delta\frac{\partial T(x_1, t)}{\partial x} = h_{m1}\left[m(x_1, t) - m_{\infty 1}\right] \tag{5c}$$

$$-D_m\frac{\partial m(x_2, t)}{\partial x} - D_m\delta\frac{\partial T(x_2, t)}{\partial x} = h_{m2}\left[m(x_2, t) - m_{\infty 2}\right] \tag{5d}$$

$$T(x, 0) = T_0 \tag{6a}$$

$$m(x, 0) = m_0 \tag{6b}$$

Eqs. (5a)–(5d) constitute the natural boundary conditions for temperature and moisture, respectively. Eqs. (5a) and (5b) represent the heat balance at $x = x_1$ and $x = x_2$. The two equations express the heat flux in terms of convection heat transfer and the phase-change energy transfer. Eqs. (5c) and (5d) represent the moisture balance at the two surfaces; the two terms on the left-hand side of the equal sign describe the supply of moisture flux under the influence of a temperature gradient and a moisture gradient, respectively. The terms to the right side of the equal sign describes the amount of moisture drawn off from or into the surfaces. Eqs. (6a) and (6b) represent initial temperature and moisture value within the domain, respectively.

3. Method of solution

Applying the Laplace transformation to Eqs. (3a), (3b), (5a)–(5d) and (6a), (6b) with respect to t, they become

$$Lx^{2n-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1-2n}\frac{\mathrm{d}\bar{T}}{\mathrm{d}x}\right)=S\bar{T}-T_0-\nu S\bar{m}+\nu m_0 \quad (7a)$$

$$Dx^{2n-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1-2n}\frac{\mathrm{d}\bar{m}}{\mathrm{d}x}\right)=S\bar{m}-m_0-\lambda S\bar{T}+\lambda T_0 \quad (7b)$$

$$k\frac{\mathrm{d}\bar{T}(x_1,S)}{\mathrm{d}x}=h_{\mathrm{C1}}\left[\bar{T}(x_1,S)-\frac{T_{\infty 1}}{S}\right]+(1-\varepsilon)h_{\mathrm{LV}}h_{\mathrm{m1}}\left[\bar{m}(x_1,S)-\frac{m_{\infty 1}}{S}\right] \quad (8a)$$

$$-k\frac{\mathrm{d}\bar{T}(x_2,S)}{\mathrm{d}x}=h_{\mathrm{C2}}\left[\bar{T}(x_2,S)-\frac{T_{\infty 2}}{S}\right]+(1-\varepsilon)h_{\mathrm{LV}}h_{\mathrm{m2}}\left[\bar{m}(x_2,S)-\frac{m_{\infty 2}}{S}\right] \quad (8b)$$

$$D_{\mathrm{m}}\frac{\mathrm{d}\bar{m}(x_1,S)}{\mathrm{d}x}+D_{\mathrm{m}}\delta\frac{\mathrm{d}\bar{T}(x_1,S)}{\mathrm{d}x}=h_{\mathrm{m1}}\left[\bar{m}(x_1,S)-\frac{m_{\infty 1}}{S}\right] \quad (8c)$$

$$-D_{\mathrm{m}}\frac{\mathrm{d}\bar{m}(x_2,S)}{\mathrm{d}x}-D_{\mathrm{m}}\delta\frac{\mathrm{d}\bar{T}(x_2,S)}{\mathrm{d}x}=h_{\mathrm{m2}}\left[\bar{m}(x_2,S)-\frac{m_{\infty 2}}{S}\right] \quad (8d)$$

$$\bar{T}(x,0)=\frac{T_0}{S} \quad (9a)$$

$$\bar{m}(x,0)=\frac{m_0}{S} \quad (9b)$$

where $\bar{T}(S)$ and $\bar{m}(S)$ are the Laplace transformation of $T(t)$ and $m(t)$, respectively; and S is Laplace transformation parameter.

Introducing a transformation function $\phi(x,S)$ such that

Table 1
The functions $\varphi_i(x,S)$, shown in Eq. (12), for slab, hollow cylinder and hollow sphere

Geometry	n	φ_i
Slab	1/2	$\mathrm{e}^{p_i x}$
Hollow cylinder	0	$I_0(p_i x)$ for $i=1,2$ $K_0(p_i x)$ for $i=3,4$
Hollow sphere	−1/2	$\mathrm{e}^{p_i x}/x$

$$\bar{T}(x,S)=\frac{\nu}{L}\phi(x,S)+\frac{T_0}{S} \quad (10a)$$

$$\bar{m}(x,S)=\frac{1}{L}\phi(x,S)-\frac{1}{S}x^{2n-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1-2n}\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)+\frac{m_0}{S} \quad (10b)$$

we find that Eq. (7a) is automatically satisfied and Eq. (7b) becomes

$$x^{2n-1}\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^{1-2n}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{2n-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1-2n}\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)\right]\right\}-\frac{S}{D}\left(1+\frac{D}{L}\right)\left[x^{2n-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1-2n}\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)\right]+\frac{S^2(1-\nu\lambda)}{\mathrm{LD}}\phi=0 \quad (11)$$

Therefore, the system of coupled ODE equations (7a) and (7b) is reduced to a single Eq. (11). Eq. (11) is a fourth-order ODE.

Assume the solution of $\phi(x,S)$ in the following form

$$\phi=\sum_{i=1}^{4}\xi_i(S)\varphi_i(x,S) \quad (12)$$

where φ_i represents different functions, as shown in Table 1, for slab, hollow cylindrical and hollow spherical geometries.

In Table 1, $p_i=\sqrt{S}q_i$, in which q_i is defined as

$$q_i=\frac{\alpha_i}{\sqrt{2D}}\left[1+\frac{D}{L}+\beta_i\sqrt{\left(1-\frac{D}{L}\right)^2+\frac{4\lambda\nu D}{L}}\right]^{1/2} \quad (13)$$

and where

$$\alpha_i=\begin{cases}1 & \text{for } i=1,2 \quad \text{if slab and hollow sphere}\\ 1 & \text{for } i=1,2,3,4 \quad \text{if hollow cylinder}\\ -1 & \text{for } i=3,4 \quad \text{if slab and hollow sphere}\end{cases}$$

and

$$\beta_i = \begin{cases} -1 & \text{for } i = 1, 3 \\ 1 & \text{for } i = 2, 4 \end{cases}.$$

The coefficients $\xi_i(S)$ ($i = 1, 2, 3, 4$) can be determined by using Eqs. (8a)–(8d). The results are then written in the matrix form

$$[\mathrm{K}]\{\xi(S)\} = \{\mathrm{Q}\} \tag{14}$$

in which

$$[\mathrm{K}] = \begin{pmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{pmatrix} \tag{15a}$$

$$\{\xi(S)\}^{\mathrm{T}} = \{\xi_1(S), \xi_2(S), \xi_3(S), \xi_4(S)\} \tag{15b}$$

$$\{Q\}^{\mathrm{T}} = \{Q_1, Q_2, Q_3, Q_4\} \tag{15c}$$

where $\{\xi(S)\}^{\mathrm{T}}$ and $\{Q\}^{\mathrm{T}}$ stand for the transpose of $\{\xi(S)\}$ and {Q}, respectively, and where

$$K_{1j} = \frac{(1-\varepsilon)h_{\mathrm{LV}}h_{\mathrm{m1}}}{S}x^{2n-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1-2n}\frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\right) + \frac{k\nu}{L}\frac{\mathrm{d}\varphi_j}{\mathrm{d}x} - \frac{1}{L}\left[h_{\mathrm{C1}}\nu + (1-\varepsilon)h_{\mathrm{LV}}h_{\mathrm{m1}}\right]\varphi_j \quad \text{at } x = x_1, j = 1, 2, 3, 4 \tag{16a}$$

$$K_{2j} = \frac{(1-\varepsilon)h_{\mathrm{LV}}h_{\mathrm{m2}}}{S}x^{2n-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1-2n}\frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\right) - \frac{k\nu}{L}\frac{\mathrm{d}\varphi_j}{\mathrm{d}x} - \frac{1}{L}\left[h_{\mathrm{C2}}\nu + (1-\varepsilon)h_{\mathrm{LV}}h_{\mathrm{m2}}\right]\varphi_j \quad \text{at } x = x_2, j = 1, 2, 3, 4 \tag{16b}$$

$$K_{3j} = -\frac{D_{\mathrm{m}}}{S}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{2n-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1-2n}\frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\right)\right] + \frac{h_{\mathrm{m1}}}{S}x^{2n-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1-2n}\frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\right) + \frac{D_{\mathrm{m}}}{L}(\delta\nu+1)\frac{\mathrm{d}\varphi_j}{\mathrm{d}x} - \frac{h_{\mathrm{m1}}}{L}\varphi_j \quad \text{at } x = x_1, j = 1, 2, 3, 4 \tag{16c}$$

$$K_{4j} = \frac{D_{\mathrm{m}}}{S}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{2n-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1-2n}\frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\right)\right] + \frac{h_{\mathrm{m2}}}{S}x^{2n-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1-2n}\frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\right) - \frac{D_{\mathrm{m}}}{L}(\delta\nu+1)\frac{\mathrm{d}\varphi_j}{\mathrm{d}x} - \frac{h_{\mathrm{m2}}}{L}\varphi_j \quad \text{at } x = x_2, j = 1, 2, 3, 4 \tag{16d}$$

$$Q_1 = \frac{1}{S}\left[h_{\mathrm{C1}}(T_0 - T_{\infty 1}) + (1-\varepsilon)h_{\mathrm{LV}}h_{\mathrm{m1}}(m_0 - m_{\infty 1})\right] \tag{16e}$$

$$Q_2 = \frac{1}{S}\left[h_{\mathrm{C2}}(T_0 - T_{\infty 2}) + (1-\varepsilon)h_{\mathrm{LV}}h_{\mathrm{m2}}(m_0 - m_{\infty 2})\right] \tag{16f}$$

$$Q_3 = \frac{h_{\mathrm{m1}}}{S}(m_0 - m_{\infty 1}) \tag{16g}$$

$$Q_4 = \frac{h_{\mathrm{m2}}}{S}(m_0 - m_{\infty 2}). \tag{16h}$$

The solution of Eq. (14) can be determined by Cramer's rule.

$$\xi_i(S) = \frac{|Z_i|}{|K|}, \quad i = 1, 2, 3, 4 \tag{17}$$

where $|Z_i|$ is the determinant of the matrix resulting from [K] in which the ith column is replaced by the column vector {Q}.

Substituting Eq. (17) into Eq. (12), Eqs. (10a) and (10b) can be rewritten as

$$\bar{T}(x, S) = \frac{\nu}{L}\sum_{i=1}^{4}\xi_i(S)\varphi_i(x, S) + \frac{T_0}{S} \tag{18a}$$

$$\bar{m}(x, S) = \sum_{i=1}^{4}\left(\frac{1}{L} - q_i^2\right)\xi_i(S)\varphi_i(x, S) + \frac{m_0}{S}. \tag{18b}$$

As it is difficult in general to find the inverse Laplace transformation of the functions $\bar{T}(x, S)$ and $\bar{m}(x, S)$ analytically, a numerical inversion method [11] may be used. This method has been proven to yield accurate results in previous research [12]. However, for some special cases, the functions at any given time can be evaluated using the inversion theorem for the Laplace transformation.

4. Analysis of a special case

As a special case of the foregoing analysis, we now consider an infinitely long hollow cylinder, having inner and outer radii x_1 and x_2, respectively, subjected to symmetrical hygrothermal loadings. The mathematical formulation is expressed as follows:

$$L\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial T}{\partial x}\right) = \frac{\partial T}{\partial t} - \nu\frac{\partial m}{\partial t} \tag{19a}$$

$$D\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial m}{\partial x}\right) = \frac{\partial m}{\partial t} - \lambda\frac{\partial T}{\partial t} \tag{19b}$$

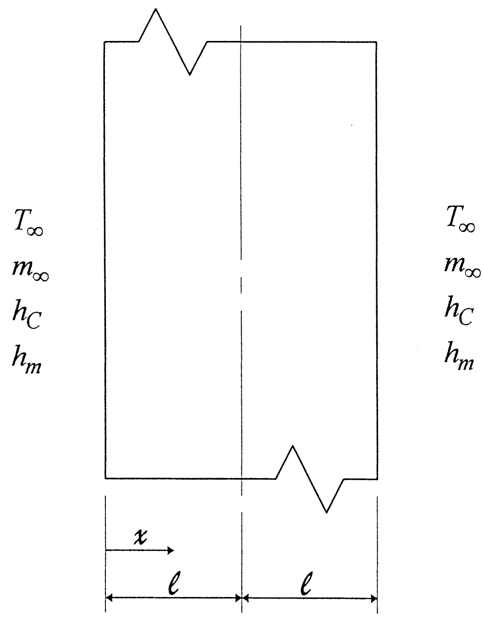

Fig. 1. Schematic representation of the wood slab.

$$T(x_1, t) = T_\infty, \qquad T(x_2, t) = T_\infty$$

$$m(x_1, t) = m_0, \qquad m(x_2, t) = m_0$$

$$T(x, 0) = T_0, \qquad m(x, 0) = m_0. \tag{20}$$

In this problem, the moisture contains vapor phase only, then the phase change within the cylinder is excluded. A similar problem have been solved in previous research [5] by using decoupling techniques.

Comparing the present problem with the previous general problem, we find that

$$h_{C1} = h_{C2} = h_C \rightarrow \infty, \quad h_{m1} = h_{m2} = h_m \rightarrow \infty, \quad p_1 = p_3, \quad p_2 = p_4,$$

$$m_{\infty 1} = m_{\infty 2} = m_\infty = m_0, \quad q_1 = q_3, \quad q_2 = q_4, \quad h_{LV} = 0,$$

$$L = \frac{k}{\rho c_p}, \quad D = \frac{k D_m}{\rho c_m (k + D_m \delta \gamma)}, \quad \nu = \frac{c_m \gamma}{c_p}, \quad \lambda = \frac{c_p D_m \delta}{c_m (k + D_m \delta \gamma)}. \tag{21}$$

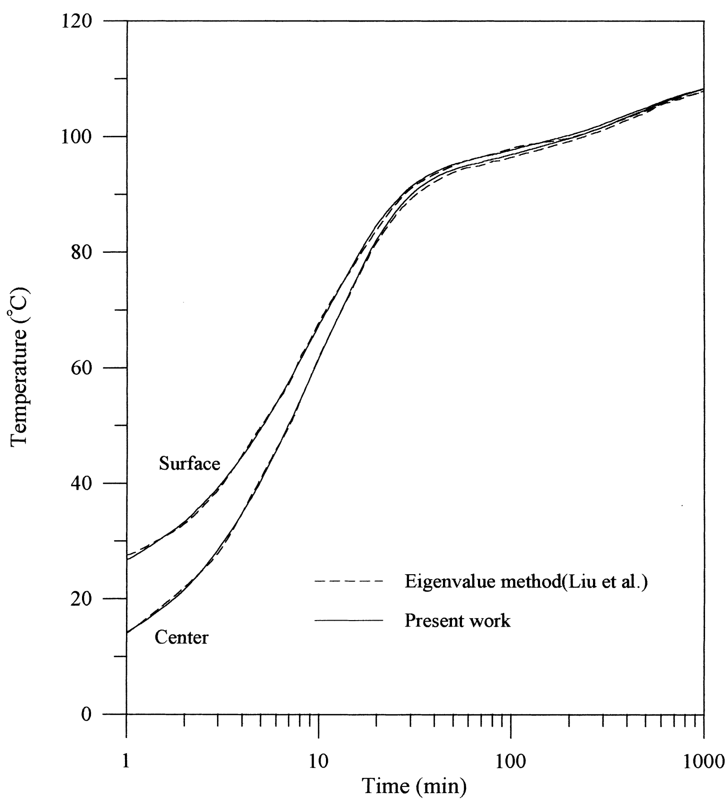

Fig. 2. Temperature at the center and surface of wood specimen for case 1.

This problem is a special case of the coupled system discussed previously. The corresponding coefficients are now reduced to

$$[\mathrm{K}] = \begin{bmatrix} \frac{v}{L} I_0(p_1 x_1) & \frac{v}{L} I_0(p_2 x_1) & \frac{v}{L} K_0(p_1 x_1) & \frac{v}{L} K_0(p_2 x_1) \\ \frac{v}{L} I_0(p_1 x_2) & \frac{v}{L} I_0(p_2 x_2) & \frac{v}{L} K_0(p_1 x_2) & \frac{v}{L} K_0(p_2 x_2) \\ \left(\frac{1}{L} - q_1^2\right) I_0(p_1 x_1) & \left(\frac{1}{L} - q_2^2\right) I_0(p_2 x_1) & \left(\frac{1}{L} - q_1^2\right) K_0(p_1 x_1) & \left(\frac{1}{L} - q_2^2\right) K_0(p_2 x_1) \\ \left(\frac{1}{L} - q_1^2\right) I_0(p_1 x_2) & \left(\frac{1}{L} - q_2^2\right) I_0(p_2 x_2) & \left(\frac{1}{L} - q_1^2\right) K_0(p_1 x_2) & \left(\frac{1}{L} - q_2^2\right) K_0(p_2 x_2) \end{bmatrix} \tag{22}$$

$$\{\mathrm{Q}\}^{\mathrm{T}} = \left[\frac{T_\infty - T_0}{S}, \quad \frac{T_\infty - T_0}{S}, \quad 0, \quad 0 \right] \tag{23}$$

Then $\xi_k(S)$ in Eq. (17) can be simplified as

$$\xi_k(S) = \frac{L(T_\infty - T_0)}{Sv(q_2^2 - q_1^2)} \left[\left(\frac{1}{L} - q_2^2\right)\gamma_k + \left(\frac{1}{L} - q_1^2\right)\eta_k \right] \cdot \frac{\left[K_0(\sqrt{S}q_k x_2) - K_0(\sqrt{S}q_k x_1)\right](\delta_{1k} + \delta_{2k}) + \left[I_0(\sqrt{S}q_k x_1) - I_0(\sqrt{S}q_k x_2)\right](\delta_{3k} + \delta_{4k})}{I_0(\sqrt{S}q_k x_1)K_0(\sqrt{S}q_k x_2) - I_0(\sqrt{S}q_k x_2)K_0(\sqrt{S}q_k x_1)}, \quad k = 1, 2, 3, 4 \tag{24}$$

where

$$\gamma_k = \begin{cases} -1 & \text{for } k = 1, 3 \\ 0 & \text{for } k = 2, 4 \end{cases}$$

and

$$\eta_k = \begin{cases} 0 & \text{for } k = 1, 3 \\ 1 & \text{for } k = 2, 4 \end{cases}$$

and δ_{ij} is the Kronecker delta which is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j \end{cases}$$

Consequently, Eqs. (18a) and (18b) become

$$\bar{T}(x, S) = \frac{T_\infty - T_0}{S(q_2^2 - q_1^2)} \sum_{n=1}^{2} \left[-\left(\frac{1}{L} - q_2^2\right)\delta_{1n} + \left(\frac{1}{L} - q_1^2\right)\delta_{2n} \right] \cdot \frac{\left[K_0(\sqrt{S}q_n x_2) - K_0(\sqrt{S}q_n x_1)\right][I]_0(\sqrt{S}q_n x) + \left[I_0(\sqrt{S}q_n x_1) - I_0(\sqrt{S}q_n x_2)\right]K_0(\sqrt{S}q_n x)}{I_0(\sqrt{S}q_n x_1)K_0(\sqrt{S}q_n x_2) - I_0(\sqrt{S}q_n x_2)K_0(\sqrt{S}q_n x_1)} + \frac{T_0}{S} \tag{25a}$$

$$\bar{m}(x, S) = \frac{L(T_\infty - T_0)\left(\frac{1}{L} - q_1^2\right)\left(\frac{1}{L} - q_2^2\right)}{Sv(q_2^2 - q_1^2)} \sum_{n=1}^{2} [-\delta_{1n} + \delta_{2n}] \cdot \frac{\left[K_0(\sqrt{S}q_n x_2) - K_0(\sqrt{S}q_n x_1)\right]I_0(\sqrt{S}q_n x) + \left[I_0(\sqrt{S}q_n x_1) - I_0(\sqrt{S}q_n x_2)\right]K_0(\sqrt{S}q_n x)}{I_0(\sqrt{S}q_n x_1)K_0(\sqrt{S}q_n x_2) - I_0(\sqrt{S}q_n x_2)K_0(\sqrt{S}q_n x_1)} + \frac{m_0}{S} \tag{25b}$$

The temperature and moisture distributions in the transformation domain, Eqs. (25a) and (25b) can be transformed to time domain by the inversion theorem for the Laplace transformation [13]. From the theorem we then have

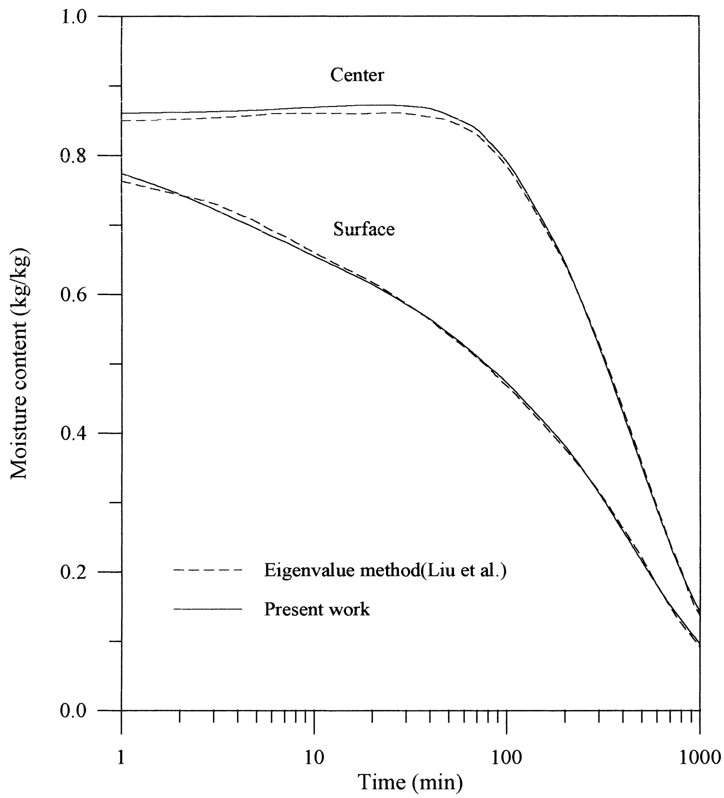

Fig. 3. Moisture content at the center and surface of wood specimen for case 1.

$$W_k(x,t) = \frac{T_\infty - T_0}{2\pi i\left(q_2^2 - q_1^2\right)} \cdot \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{1}{z} \cdot \left\{ -\left(\frac{1}{L} - q_2^2\right) \times \left[\delta_{1k} + \frac{L}{v}\left(\frac{1}{L} - q_1^2\right)\delta_{2k}\right]\frac{U_1}{V_1} + \left(\frac{1}{L} - q_1^2\right)\left[\delta_{1k} + \frac{L}{v}\left(\frac{1}{L} - q_2^2\right)\delta_{2k}\right]\frac{U_2}{V_2}\right\} \mathrm{e}^{zt}\mathrm{d}z + T_0\delta_{1k} + m_0\delta_{2k} \tag{26}$$

where $k = 1, 2$; W_1 and W_2 represent T and m, respectively; and

$$U_k = \left\{\left[K_0\left(\sqrt{z}q_k x_2\right) - K_0\left(\sqrt{z}q_k x_1\right)\right] I_0\left(\sqrt{z}q_k x\right) + \left[I_0\left(\sqrt{z}q_k x_1\right) - I_0\left(\sqrt{z}q_k x_2\right)\right] K_0\left(\sqrt{z}q_k x\right)\right\} \mathrm{e}^{zt}, \quad k = 1, 2 \tag{27a}$$

$$V_k = I_0\left(\sqrt{z}q_k x_1\right) K_0\left(\sqrt{z}q_k x_2\right) - I_0\left(\sqrt{z}q_k x_2\right) K_0\left(\sqrt{z}q_k x_1\right), \quad k = 1, 2 \tag{27b}$$

The singularities of the integrand are $z = 0$, $\sqrt{z}q_1 = i\alpha_n$, and $\sqrt{z}q_2 = i\alpha_n$, which correspond to the simple poles $z = 0$, $z = -\alpha_n^2/q_1^2$, $z = -\alpha_n^2/q_2^2$, respectively, where α_n satisfies

$$J_0(\alpha_n x_1) Y_0(\alpha_n x_2) - J_0(\alpha_n x_2) Y_0(\alpha_n x_1) = 0 \tag{28}$$

Note that the relationships $I_0(i\alpha_n x) = J_0(\alpha_n x)$ and $K_0(i\alpha_n x) = \frac{-\pi i}{2}[J_0(\alpha_n x) - iY_0(\alpha_n x)]$ are used to obtain the above equation.

Using the residue theorem to evaluate Eq. (26), yields

$$W_k(x,t) = (T_\infty - T_0)\delta_{1k} \cdot \mathrm{Res}(0) + (T_\infty - T_0)\frac{-\left(\frac{1}{L} - q_2^2\right)}{q_2^2 - q_1^2} \times \left[\delta_{1k} + \frac{L}{v}\left(\frac{1}{L} - q_1^2\right)\delta_{2k}\right] \cdot \sum_{n=1}^{\infty} \mathrm{Res}\left(-\frac{\alpha_n^2}{q_1^2}\right) + (T_\infty - T_0)\frac{\frac{1}{L} - q_1^2}{q_2^2 - q_1^2} \times \left[\delta_{1k} + \frac{L}{v}\left(\frac{1}{L} - q_2^2\right)\delta_{2k}\right] \cdot \sum_{n=1}^{\infty} \mathrm{Res}\left(-\frac{\alpha_n^2}{q_2^2}\right) + T_0\delta_{1k} + m_0\delta_{2k} \quad k = 1, 2. \tag{29}$$

The residue at $z = 0$ can be evaluated for small arguments $I_0(z) \approx 1$, $I_1(z) \approx z/2$, $K_0(z) \approx -\ln z$, and $K_1(z) \approx 1/z$; then the limit can be taken by letting $z \to 0$, that is

$$\mathrm{Res}(0) = 1. \tag{30}$$

The residue at $z = -\alpha_n^2/q_1^2$, and $z = -\alpha_n^2/q_2^2$ may be obtained from

$$\mathrm{Res}\left(-\frac{\alpha_n^2}{q_k^2}\right) = \frac{U_k\left(-\alpha_n^2/q_k^2\right)}{-\alpha_n^2/q_k^2 (\mathrm{d}V_k/\mathrm{d}z)_{z=-\alpha_n^2/q_k^2}}, \quad k = 1, 2. \tag{31}$$

Differentiating Eq. (27b), we have

$$\frac{\mathrm{d}V_k}{\mathrm{d}z} = \frac{q_k}{2z^{1/2}}\left[x_1 I_1\left(\sqrt{z}q_k x_1\right) K_0\left(\sqrt{z}q_k x_2\right) - x_2 I_0\left(\sqrt{z}q_k x_1\right) K_1\left(\sqrt{z}q_k x_2\right) - x_2 I_1\left(\sqrt{z}q_k x_2\right) K_0\left(\sqrt{z}q_k x_1\right) + x_1 I_0\left(\sqrt{z}q_k x_2\right) K_1\left(\sqrt{z}q_k x_1\right)\right] \tag{32}$$

In Eq. (32), inserting $i\alpha_n$ in place of $\sqrt{z}q_k$, and using the relationships $I_0(z)K_1(z) + I_1(z)K_0(z) = 1/z$, $I_0(i\alpha_n x) = J_0(\alpha_n x)$, and $V_k = 0$ in Eq. (27b), the following form is obtained

$$-\frac{\alpha_n^2}{q_k^2}\frac{\mathrm{d}V_k}{\mathrm{d}z}\bigg|_{z=-\alpha_n^2/q_k^2} = \frac{J_0^2(\alpha_n x_2) - J_0^2(\alpha_n x_1)}{2J_0(\alpha_n x_1)J_0(\alpha_n x_2)}, \quad k = 1, 2. \tag{33}$$

Similarly,

$$U_k\left(-\frac{\alpha_n^2}{q_k^2}\right) = -\frac{\pi}{2}\left\{\left[Y_0(\alpha_n x_2) - Y_0(\alpha_n x_1)\right]J_0(\alpha_n x) + \left[J_0(\alpha_n x_1) - J_0(\alpha_n x_2)\right]Y_0(\alpha_n x)\right\} \times \mathrm{e}^{-(\alpha_n^2/q_k^2)t}, \quad k = 1, 2 \tag{34}$$

Finally, substituting Eqs. (33) and (34) into Eq. (31), and then substituting Eqs. (31) and (30) into Eq. (29), the following results are obtained:

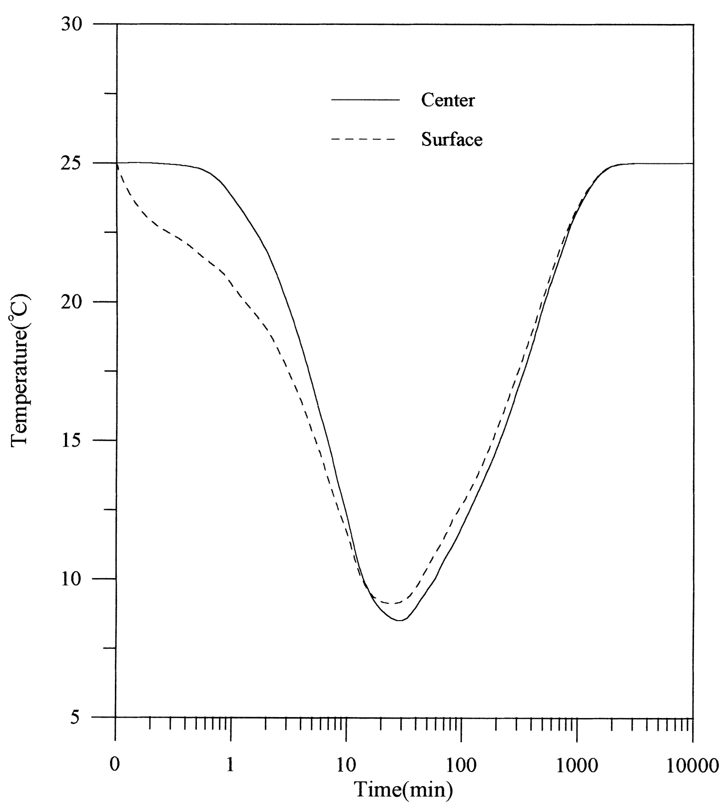

Fig. 4. Temperature at the center and surface of wood specimen for case 2.

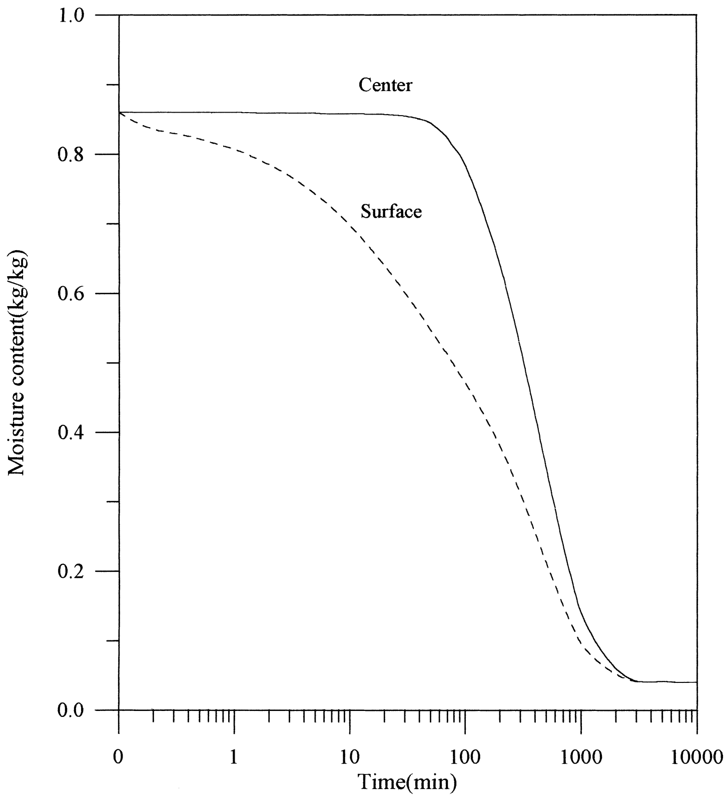

Fig. 5. Moisture content at the center and surface of wood specimen for case 2.

$$W_k(x,t) = (T_\infty - T_0)\delta_{1k} - \pi(T_\infty - T_0)\sum_{n=1}^{\infty}\frac{\left[J_0(\alpha_n x_1) - J_0(\alpha_n x_2)\right]\left[J_0(\alpha_n x)Y_0(\alpha_n x_2) - J_0(\alpha_n x_2)Y_0(\alpha_n x)\right]}{J_0^2(\alpha_n x_1) - J_0^2(\alpha_n x_2)}\cdot J_0(\alpha_n x_1)$$
$$\cdot\left\{\frac{-\left(\frac{1}{L} - q_2^2\right)}{q_2^2 - q_1^2}\left[\delta_{1k} + \frac{L}{\nu}\left(\frac{1}{L} - q_1^2\right)\delta_{2k}\right]e^{-(\alpha_n^2/q_1^2)t} + \frac{\frac{1}{L} - q_1^2}{q_2^2 - q_1^2}\left[\delta_{1k} + \frac{L}{\nu}\left(\frac{1}{L} - q_2^2\right)\delta_{2k}\right]e^{-(\alpha_n^2/q_2^2)t}\right\} + T_0\delta_{1k} \quad (35)$$
$$+ m_0\delta_{2k}$$

$k = 1, 2$

In order to compare the results written in Eq. (35) with the results of [5], Eq. (35) can be rewritten as follows:

$$\frac{T(x,t) - T_0}{T_\infty - T_0} = 1 - \frac{\pi}{B_1 + B_2}\sum_{n=1}^{\infty}\frac{\left[J_0(\alpha_n x_1) - J_0(\alpha_n x_2)\right]J_0(\alpha_n x_1)U_0(\alpha_n x)}{J_0^2(\alpha_n x_1) - J_0^2(\alpha_n x_2)}\times\left(B_1\, e^{-\alpha_n^2 t/q_1^2} + B_2\, e^{-\alpha_n^2 t/q_2^2}\right) \quad (36a)$$

$$\frac{m(x,t) - m_0}{\lambda(T_\infty - T_0)} = \frac{\pi L B_1 B_2}{D(B_1 + B_2)}\sum_{n=1}^{\infty}\frac{\left[J_0(\alpha_n x_1) - J_0(\alpha_n x_2)\right]J_0(\alpha_n x_1)U_0(\alpha_n x)}{J_0^2(\alpha_n x_1) - J_0^2(\alpha_n x_2)}\times\left(-e^{-\alpha_n^2 t/q_1^2} + e^{-\alpha_n^2 t/q_2^2}\right) \quad (36b)$$

where

$$U_0(\alpha_n x) = J_0(\alpha_n x)Y_0(\alpha_n x_2) - J_0(\alpha_n x_2)Y_0(\alpha_n x) \quad (37a)$$

$$B_1 = \frac{1 - Dq_1^2}{\nu\lambda} \tag{37b}$$

$$B_2 = \frac{Dq_2^2 - 1}{\nu\lambda} \tag{37c}$$

and where $(T - T_0)/(T_\infty - T_0)$ and $(m - m_0)/\lambda(T_\infty - T_0)$ are dimensionless forms of T and m, respectively.

The temperature and moisture distributions, as written in Eqs. (36a) and (36b), are identical to those obtained by previous research [5].

5. Numerical results and discussions

Now we consider a wood slab, subjected to symmetrical hygrothermal loadings; the heat and moisture transfer are coupled within the slab and its boundaries. Two cases are discussed here. For case 1, the coupled system is modeled by Eqs. (3a), (3b) to (6a), (6b). The geometry and the material properties of the wood slab shown in Fig. 1, which were used by Liu et al. [3], are used in the numerical calculations for comparison. Therefore, absorption heat in Eqs. (3a) and (3b) is assumed to be negligible and the geometry and boundary conditions in Eqs. (5a)–(5d) are assumed to be symmetric. The results can be obtained from Eqs. (18a) and (18b), and the data are as follows:

$$n = 1/2, \quad T_0 = 10°\text{C}, \quad T_\infty = 110°\text{C}, \quad m_0 = 86°\text{M},$$

$$m_\infty = 4°\text{M}, \rho = 370\ \text{kg/m}^3, \quad k = 0.65\ \text{W/(m K)},$$

$$D_\text{m} = 2.2 \times 10^{-8}\ \text{kg/(m s °M)},$$

$$h_\text{LV} = 2500\ \text{kJ/kg}, \quad h_\text{C1} = h_\text{C2} = h_\text{C} = 2.25\ \text{W/(m}^2\ \text{K)},$$

$$\varepsilon = 0.3, \quad c_\text{p} = 2500\ \text{J/(kg K)}, \quad c_\text{m} = 0.01\ \text{kg/(kg °M)},$$

$$\delta = 2.0°\text{M/K}, \quad h_\text{m1} = h_\text{m2} = h_\text{m} = 2.5 \times 10^{-6}\ \text{kg/(m}^2$$

$$\text{s °M)}, \quad \ell = 0.012\ \text{m}, \quad \gamma = 0.$$

For case 1, the temperature and moisture evolution at the surface and at the middle of the slab are shown in Figs. 2 and 3. It can be seen that the results from the two different methods agree well, even though the initial moisture content deviates a little, as shown in Fig. 3. The discrepancies might be caused by not including enough complex eigenvalues when using the eigenvalue method [3,4].

For case 2, to evaluate the ambient temperature effects on the moisture of the medium during drying, we make $T_0 = T_\infty = 25°\text{C}$, and the other data are kept the same as for case 1. Fig. 4 shows the temperature evolution at the surface and at the center of the slab. The temperature of the slab at initial time is equal to that of the atmosphere, while the moisture of the slab is higher than that of the atmosphere, so that moisture is transferred from the surface of the slab as vapor to the atmosphere. Initially, this evaporation causes cooling of the slab at the surface. Subsequently, the temperature within the slab also decreases due to heat conduction. After the temperature decreases to a value of about 9°C, it begins to increase due to heat convection from the surfaces. Simultaneously, the slab gains heat from the ambient atmosphere and loses heat through latent heat of evaporation. According to our mathematical analysis, the slab should reach an equilibrium temperature at $T = 25°\text{C}$ after 30 h.

The moisture content evolution is shown in Fig. 5. As expected, the moisture content of the slab decreases with increases in time, until it reaches an equilibrium value of 0.04 kg/kg after 30 h. The moisture content at the surfaces is lower than in the interior.

Comparison of Figs. 3 and 5 indicates that the temperature has some effects on moisture migration during drying. Generally, the coupling effect between the temperature and moisture content becomes significant as the value D/L approaches unity, while it will diminish as $\nu \cdot \lambda$ approaches zero. In this study, according to the data given by Liu et al., they are $D/L \approx 1/12$ and $\nu \cdot \lambda \approx 1/20$.

6. Conclusion

This study proposes a new analytical approach that consists of applying the Laplace transform technique and a transformation function to solve the problem of coupled temperature and moisture transport. The results, shown to be the same as published analytical solutions obtained using a decoupled technique, compare very well against published analytical solution for a wood slab configuration, and can serve to evaluate the accuracy of approximate or numerical solutions. Therefore, the method is recommended for analytically solving problems involving coupled temperature and moisture transport in porous materials.

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